Pergamon

S0021-8928(96)00029-9

J. Appl. Maths Mechs, Vol. 60, No. 2, pp. 227–232, 1996

0021-8928/96 $24.00+0.00

ON THE THEORY OF LINEAR GYROSCOPIC SYSTEMS†

A. A. ZEVIN

Dnepropetrovsk

(Received 23 February 1995)

The behaviour of the oscillation frequencies of Hamiltonian systems when their stiffness and inertia are changed is reviewed. The number of frequencies of the first and second kind, expressed in terms of the number of positive and negative eigenvalues of the Hamiltonian, is found. It follows from these results, in particular, that Rayleigh's classical theorem only holds for gyroscopic systems when the number of its frequencies is equal to the number of frequencies of the system without gyroscopic forces. The degree of instability of a gyroscopic system when there are small dissipative forces is found; in a gyroscopically stabilized system, it is half the degree of static instability.

1. Consider the linear gyroscopic system

$$M\ddot{\mathbf{x}} + G\dot{\mathbf{x}} + C\mathbf{x} = 0 \tag{1.1}$$

where $\mathbf{x} \in R^n$, M and C are symmetric inertia and stiffness matrices, G is a skew symmetric gyroscopic force matrix and the matrix M is assumed to be positive definite ($M > 0$).

Let $r_0 \leqslant n$ be the number of positive eigenvalues of the matrix C. Then, when there are no gyroscopic forces ($G = 0$) the system has r_0 vibrational frequencies ω_k^0. By Rayleigh's theorem [1], the frequencies increase when there is an increase in the stiffness or a decrease in the inertia (that is, when there is an increase in the potential energy and a decrease in the kinetic energy). Rayleigh proved this theorem using perturbation theory: another well-known proof based on the minimax property of eigenvalues is due to Courant [2].

We shall present some results of an analysis and extension of this remarkable theorem. First, we note that the symmetry condition for the matrices M and C is essential; otherwise, as has been shown in [3], for any frequency ω_k^0, an increase in C or a decrease in M can be found such that ω_k^0 decreases. Rayleigh's theorem cannot therefore be extended to systems with non-conservational positional forces. It is easy to show that it is also necessary for the matrix M to be positive definite.

The first attempt to extend Rayleigh's theorem to systems with gyroscopic forces [4] led to conclusions containing an error, which remained unnoticed in spite of the fact that the results continued to be used and discussed (for example, see [5]). We shall therefore explain the method proposed in [4].

Let $i\omega_k$ be an imaginary root of the characteristic equation and let $\mathbf{x}_k$ be the corresponding eigenvector, that is,

$$(-\omega_k^2 M + i\omega_k G + C)\mathbf{x}_k = 0 \tag{1.2}$$

We find by taking the scalar product of equality (1.2) and $\mathbf{x}_k$ that ω_k satisfies the quadratic equation

$$m_k\omega^2 - g_k\omega - c_k = 0 \tag{1.3}$$

where $m_k = (M\mathbf{x}_k, \mathbf{x}_k)$, $g_k = (iG\mathbf{x}_k, \mathbf{x}_k)$ and $c_k = (C\mathbf{x}_k, \mathbf{x}_k)$; $(\mathbf{a}, \mathbf{b})$ denotes the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

Since M and C are real symmetric matrices and the matrix iG is Hermitian, then m_k, g_k and c_k are real numbers and, moreover, $m_k > 0$ by virtue of the fact that $M > 0$. When account is taken of the fact that the roots of the characteristic equations are complex conjugates, it can be assumed that $\omega_k > 0$, that is, ω_k is identical with the positive root of Eq. (1.3). It can be shown that ω_k has a stationary value with respect to the components of the vector $\mathbf{x}_k$ and the latter therefore cannot vary when there are variations in stiffness and inertia.

We shall initially assume that the system is statically stable ($C > 0$). The roots ω' and ω'' of Eq. (1.3) are then of opposite sign. Since $-\omega'\omega''c_k/m_k$ increases with respect to c_k, and $\omega' + \omega'' = g_k/m_k$ does not

†*Prikl. Mat. Mekh.* Vol. 60, No. 2, pp. 233–239, 1996.

depend on c_k, the positive root $\omega' = \omega_k$ increases as c_k increases, that is, the stiffness increases. It can be shown in a similar manner than ω_k decreases as the inertia increases.

We shall now assume that the system is statically unstable (there are negative eigenvalues among the eigenvalues of the matrix C). If, in this case, $g_k > 0$, $c_k < 0$ and $g_k^2 + 4c_k m_k > 0$ for a certain vector $\mathbf{x}_k$, then both of the roots ω' and ω'' are positive. It can be shown, using the above arguments that, for an increase in the stiffness and a decrease in the inertia, the larger root increases and the smaller root decreases. Since it is unknown "a priori" to which of these roots a frequency ω_k corresponds, it is impossible to draw any conclusion regarding its behaviour. This fact was not taken into account in [4] and the assertion which was made concerning the increase in the frequencies when the stiffness is increased is unfounded. We shall show that it is untrue in the general case.

Let us now assume that the matrix C has p negative eigenvalues, the remaining eigenvalues being positive. We know [6] that, in the case of even p, the matrix G can be chosen in such a manner that all the roots of the characteristic equation

$$\det\left\| M\lambda^2 + G\lambda + C \right\| = 0 \tag{1.4}$$

are pure imaginary ($\lambda_k = \pm i\omega_k$, $k = 1, \dots, n$), that is, there is gyroscopic stabilization. We shall now show that, in such a system, certain frequencies can decrease when C increases.

Without loss of generality, we shall assume that M is the identity matrix and that C is a diagonal matrix ($C = \operatorname{diag}[c_1, \dots, c_n]$) and, moreover, that $c_k < 0$ when $k \leqslant p$ and $c_k > 0$ when $k > p$.

The free term in Eq. (1.4) is then equal to

$$a_0 = \omega_1^2\omega_2^2 \dots \omega_n^2 = \det C = c_1 c_2 \dots c_n \tag{1.5}$$

We put $C(\mu) = \operatorname{diag}[c_1 + \mu, c_2, \dots, c_n]$ and then $da_0(\mu)/d\mu = c_2 \dots c_n$. In this product an odd number $(p-1)$ of factors c_k are negative while the remaining factors are positive. Hence, $da_0(\mu)/d\mu < 0$ and, consequently, $d\omega_k^2(\mu)/d\mu < 0$ for a certain k. The matrix $dC(\mu)/d\mu = \operatorname{diag}[1, 0, \dots, 0]$ is negative definite: it is clear that it can be made positive definite by means of a perturbation which may be as small as desired but which does not violate the inequality. Hence, when there is an increase in stiffness, individual frequencies of a gyroscopically stabilized system can decrease. It is shown below that such frequencies always exist.

It has been shown by a rigorous solution of a problem in [7] that Rayleigh's theorem can be extended to gyroscopic systems with a non-negative definite stiffness matrix ($C \geqslant 0$). Another proof of this assertion has subsequently been obtained [8].

We shall now consider the more general Hamiltonian system

$$J\dot{\mathbf{y}} = A\mathbf{y} \tag{1.6}$$

where $\mathbf{y} \in R^{2n}$, J is a non-singular skew symmetric matrix and A is a symmetric matrix.

We know that, by making the substitution

$$\mathbf{x} = \mathbf{q}, \quad M\dot{\mathbf{x}} = \mathbf{p}, \quad \mathbf{y} = [\mathbf{q}, \mathbf{p}] \tag{1.7}$$

Equation (1.1) can be reduced to the form of (1.6) with

$$J = \left\| \begin{matrix} -G & -I_n \\ I_n & 0 \end{matrix} \right\|, \quad A = \left\| \begin{matrix} C & 0 \\ 0 & M^{-1} \end{matrix} \right\| \tag{1.8}$$

where I_n is the identity matrix of order n. We note that, if the matrix M decreases, then M^{-1} increases [9] and, hence, A increases when C increases and M decreases.

In (1.6), let $A = A(\mu) = A_0 + \mu A_1$ and $A_1 > 0$ (that is, $A(\mu)$ increases with respect to μ). If λ_k is a simple root of the characteristic equation when $\mu = 0$, then $\lambda_k(\mu)$ is an analytic function. The standard perturbation method procedure gives [10]

$$\left.\frac{d\lambda_k(\mu)}{d\mu}\right|_{\mu=0} = \frac{(A_1\mathbf{y}_k, \mathbf{y}_k)}{(J\mathbf{y}_k, \mathbf{y}_k)} \tag{1.9}$$

where $\mathbf{y}_k = [\mathbf{q}_k, \mathbf{p}_k]$ is the eigenvector corresponding to the eigenvalue λ_k ($A_0\mathbf{y}_k = \lambda_k J\mathbf{y}_k$). If the root

λ_k is multiple but simple elementary divisors of the matrix $J^{-1}A_0$ correspond to it, then, by an appropriate choice of the vectors $\mathbf{y}_k$, the values of $d\gamma_k(\mu)/d\mu$ when $\mu = 0$ are also determined by expressions (1.9).

Let $\lambda_k = i\omega_k$. Then $\lambda_k(\mu) = i\omega_k(\mu)$. Since the numerator in (1.9) is positive ($A_1 > 0$), the sign of $d\omega_k(\mu)/d\mu$ when $\mu = 0$ is identical with the sign of $l_k = i(J\mathbf{y}_k, \mathbf{y}_k) = (A_0\mathbf{y}_k, \mathbf{y}_k)/\omega_k$.

If $A_0 > 0$, all the roots are pure imaginary and all the elementary divisors of the matrix $J^{-1}A_0$ are simple [10]. All the frequencies ω_k therefore increase when A_0 increases. Hence, Rayleigh's theorem holds for Hamiltonian systems with a positive definite Hamiltonian. Conversely, if $A_0 < 0$, all the frequencies decrease when A_0 increases.

In accordance with the classification introduced in [11], the vibrational frequencies ω_k of a Hamiltonian system for which $l_k > 0$ and $l_k < 0$ are referred to as frequencies of the first and second kind, respectively. Hence, when the Hamiltonian becomes larger, frequencies of the first kind increase while frequencies of the second kind decrease.

A Hamiltonian system with periodic coefficients

$$J\dot{\mathbf{x}} = A(\omega t, \mu)\mathbf{x}, \qquad (1.10)$$
$$A = A(\omega t, \mu) = A_0(\omega t) + \mu A_1(\omega t)$$

has been considered in [12]. In this system, the symmetric positive definite matrices $A_0(\omega t)$ and $A_1(\omega t)$ are periodic with respect to t. It was proved that the critical frequencies of parametric resonance $\omega_p(\mu)$, $p = 1, 2, \ldots$ (which correspond to the limits of the domains of stability of Eq. (1.10)) increase with respect to μ; this result extends Rayleigh's theorem to parametrically excited systems.

In this paper, we formulate the problem of the behaviour of free vibrations of the non-linear Hamiltonian system

$$J\dot{\mathbf{x}} = H_{\mathbf{x}}(\mathbf{x}, \mu) \qquad (1.11)$$

when the Hamiltonian H is perturbed. It is assumed here that the Hessian $H_{\mathbf{xx}}(\mathbf{x}, \mu) = \| \partial^2 H/\partial x_p \partial x_k \|_1^{2n} > 0$ and $\partial H_{\mathbf{xx}}(\mathbf{x}, \mu)/\partial\mu > 0$ (these inequalities are analogous to the conditions $A_0 > 0$ and $A_1 > 0$ in the linear case). Since, here, the oscillation frequencies depend on the total energy h, the latter was fixed ($H(\mathbf{x}, \mu) = h$). It was found that, in the general case, the frequency $\omega_k(h, \mu)$ cannot increase with respect to μ even in a system with a single degree of freedom. However, the inequality $\partial\omega_k(h, \mu)/\partial\mu > 0$ holds for values of h at which $\omega_k(h, \mu)$ is stationary with respect to $h(\partial\omega_k(h, \mu)/\partial h = 0$, that is, an assertion analogous to Rayleigh's theorem holds.

2. We shall now investigate the behaviour of the oscillation frequencies of system (1.11) when there is a change in the stiffness and inertia. As was noted above, the extension of Rayleigh's theorem to such systems holds provided that the system is statically stable while, when there are no gyroscopic forces, this assumption is unnecessary. This is associated with the fact that, when $G = 0$, all the frequencies are of the first kind.

Actually, on taking account of the fact that, here, $C\mathbf{x}_k = \omega_k^2 M\mathbf{x}_k$, $\mathbf{p}_k = i\omega_k M\mathbf{x}_k$ and $M > 0$, we find

$$l_k = (A\mathbf{y}_k, \mathbf{y}_k)/\omega_k = [(C\mathbf{x}_k, \mathbf{x}_k) + (M^{-1}\mathbf{p}_k, \mathbf{p}_k)]/\omega_k = 2\omega_k(M\mathbf{x}_k, \mathbf{x}_k) > 0 \qquad (2.1)$$

The following theorem establishes the numbers of frequencies n_1 and n_2 of the first and second kinds of the Hamiltonian system (1.6) with a non-singular matrix A. We note that existing results of this type (due to Kovalenko and Krein [11]) refer to a stable system (the total number of frequencies of which $s = n$) and amount to the following: the number r of positive eigenvalues of the matrix A is even, $n_1 = r/2$ and $n_2 = n - n_1$.

By virtue of the symmetry of the matrix A its eigenvalues are real and, since $\det A \neq 0$, $2n - r$ eigenvalues are negative.

Theorem 1. The number of oscillation frequencies of system (1.6) satisfies the inequality $s \geqslant |r - n|$ and the quantity $s - |r - n|$ is even: $(s + r - n)/2$ frequencies are of the first kind and $(s - r + n)/2$ frequencies are of the second kind.

Proof. We know [10] that non-singular matrices S and D exist such that

$$J = S^T J' S, \quad A = D^T A' D \tag{2.2}$$

where J is any non-singular matrix which has the same number of positive, negative and zero eigenvalues as A. The superscript T denotes transposition.

Using (2.2), the substitution $\mathbf{y} = S\mathbf{z}$ reduces (16) to the form

$$J' \dot{\mathbf{z}} = R^T A' R\mathbf{z}, \quad R = DS \tag{2.3}$$

We shall assume that

$$J' = \operatorname{diag}(J_2, \ldots, J_2), \quad J_2 = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix} \tag{2.4}$$

and that A' is a diagonal matrix with elements 1 and –1, the total number of which are equal to r and $2n - r$, respectively.

We connect the matrix R by a continuous curve with the unit matrix I_{2n}, that is, we put $R = R(\varepsilon)$, where $R(\varepsilon)$ is a non-singular matrix such that $R(0) = I_{2n}$, $R(1) = R$. When $\varepsilon = 0$, system (2.3) splits into n systems of the form

$$J_2 \dot{\mathbf{z}}_k = \begin{Vmatrix} a_k & 0 \\ 0 & b_k \end{Vmatrix} \mathbf{z}_k \tag{2.5}$$

where a_k and b_k take the values 1 and –1. If $a_k = b_k = 1$ or $a_k = b_k - 1$, then a frequency $\omega_k = 1$ of the first or second kind, respectively, corresponds to Eq. (2.5), otherwise, the roots of the characteristic equation are equal to ± 1. It is obvious that the minimum number of pairs $a_k = b_k$ is equal to $|r - n|$ and that the corresponding frequencies are of the first ($r > n$) or second ($r < n$) kind.

Assuming that, when $\varepsilon = 0$, the number of frequencies is equal to $|r - n|$, we now consider the behaviour of the system when the parameter ε increases in the interval [0, 1]. Since system (2.3) is canonical, a root $-\alpha_p(\varepsilon) + i\omega_p(\varepsilon)$ exists together with the root $\alpha_p(\varepsilon) + i\omega_p(\varepsilon)$. Hence, multiple roots can only converge with the imaginary axis and then certainly remain on the axis if the corresponding frequencies are of the first kind [13]. Since the matrix $R^T(\varepsilon)A'R(\varepsilon)$ is non-singular, $\omega_k(\varepsilon) \neq 0$ and the number of frequencies therefore does not change until a certain value of $\alpha_p(\varepsilon)$ vanishes. In this case, a double frequency (or, in the general case, a $2m$-degenerate frequency) "consisting" of an equal number of frequencies of the first and second kind appears. When ε increases further, these frequencies diverge, generally speaking. In a similar way, only imaginary roots which correspond to frequencies of a different kind can converge with the imaginary axis after meeting it. It follows from these considerations that the number of frequencies s cannot be less than $|r - n|$ and the number of additional frequencies $s - |r - n|$ is odd. Moreover, half of these are of the first kind and half are of the second kind. Consequently, the number of frequencies of the first and second kinds is equal to $(s + r - n)/2$ and $(s - r + n)/2$. The theorem is proved.

Taking into account the fact that $s \leqslant n$ for even r and $s \leqslant n - 1$ for odd r, we find that the number of frequencies of the first and second kinds satisfies the inequalities

$$r - n \leqslant n_1 \leqslant [r/2], \quad n - r \leqslant n_2 \leqslant [(2n - r)/2] \tag{2.6}$$

where $[a]$ is the integer part of a.

It can be seen from the proof that the theorem also holds in the case of a non-singular complex-valued Hermitian matrix A.

We shall illustrate the theorem using the example of the gyroscopic system (1.1). Since $M > 0$, the number of positive eigenvalues of the matrix A is equal to $r = n + r_0$, where r_0 is the number of positive eigenvalues of the matrix C. Consequently, of the s frequencies of system (1.1), $(s + r_0)/2$ are of the first kind and $(s - r_0)/2$ are of the second kind. In accordance with formula (1.9), when the matrix A increases (that is, when C increases and M decreases) $(s + r_0)/2$ frequencies of system (1.1) increase and $(s - r_0)/2$ frequencies decrease. We note that the vectors $\mathbf{x}_k$ and $\mathbf{p}_k = i\omega_k M\mathbf{x}_k$ are non-zero vectors and, therefore, the indicated change in the frequencies also holds if only matrix C or matrix M changes.

The following condition for Rayleigh's theorem to hold in the case of system (1.1) ensues from the result obtained.

Corollary. Rayleigh's theorem only holds for a gyroscopic system when the number of its frequencies is equal to the number of frequencies of the system when there are no gyroscopic forces acting.

This condition is certainly satisfied if the system is statically stable ($C > 0$, $s = r_0 = n$) and, also, if the degree of instability (the number of negative eigenvalues of the matrix C) is equal to one (here, $s = r_0 = n - 1$).

We shall now assume that the matrix C (and, consequently, the corresponding matrix A) is singular. The number of zero roots of the characteristic equation is independent of the matrices M and G and is equal to the number l of zero eigenvalues of the matrix C. Hence, as G increases from zero up to the specified value, the frequencies $\omega_k(\varepsilon)$ do not vanish. Therefore, the arguments used in the proof of the theorem remain the same. Allowing for the fact that the matrix A has $n + r_0$ positive and $n - r_0 - l$ negative eigenvalues, it is possible to calculate the minimum and maximum number of pairs $a_k = b_k \neq 0$ and thereby find the lower and upper limits of the number of frequencies of the system

$$r_0 \leqslant s \leqslant [(n + r_0)/2] + [(n - r_0 - l)/2] \tag{2.7}$$

Here, as in the case of a non-singular matrix C, there are $(s + r_0)/2$ frequencies of the first kind and $(s - r_0)/2$ frequencies of the second kind.

When the matrix C increases, the zero eigenvalues become positive and the number of frequencies therefore increases by l and, moreover, they are all of the first kind.

3. We shall now consider the gyroscopic system with dissipative forces

$$M\ddot{\mathbf{x}} + \mu F\dot{\mathbf{x}} + G\dot{\mathbf{x}} + C\mathbf{x} = 0 \tag{3.1}$$

where F is a symmetric, positive definite matrix and μ is a small parameter.

We shall assume that the system is statically unstable. When $G = 0$, the number of roots of Eq. (1.4) with a positive real part (and, consequently, the number of solutions that increase without limit) is equal to the degree of instability $n - r_0$. If the latter is even, then, for certain gyroscopic forces, the system becomes stable. However, by Thomson and Tait's fourth theorem [6], the introduction of dissipative forces, which can be as small as desired, destroys this stability. Naturally, the degree of instability of the resulting system is determined by the number of roots of the characteristic equation with a positive real part. The following theorem shows that this quantity is independent of the actual form of the dissipative forces and is equal to the degree of static instability.

Theorem. When small dissipative forces are introduced, the degree of instability of a gyroscopically stabilized system becomes equal to $n - r_0$.

Proof. Since, according to our assumption, the system is stable when $\mu = 0$, all the roots of the characteristic equation (1.2) are imaginary and, in the case of multiple roots, simple elementary divisors correspond to them.

Substitution of (1.7) reduces Eq. (3.1) to the form

$$J\dot{\mathbf{y}} = (A_0 + \mu A_1)\mathbf{y}, \quad A_1 = \begin{Vmatrix} 0 & FM^{-1} \\ 0 & 0 \end{Vmatrix}$$

(J and A are determined using formulae (1.8)).

Let $\mathbf{y}_k = [\mathbf{x}_k, i\omega_k M\mathbf{x}_k]$ be the eigenvector which corresponds to the root $i\omega_k$. Formula (1.9) also holds in the case of the asymmetric matrix A_1, and therefore

$$d\lambda_k(\mu)/d\mu\big|_{\mu=0} = -\omega_k(F\mathbf{x}_k, \mathbf{x}_k)/l_k$$

Since $F > 0$, this quantity is positive if the frequency ω_k is of the second kind ($l_k < 0$). Allowing for the fact that the roots of Eq. (3.1) are complex conjugates, we find that, for small μ, the number of roots with a positive real part is equal to twice the number of frequencies of the second kind, that is, $n - r_0$ ($s = n$ in a gyroscopically stabilized system). The theorem is proved.

It can be seen from the proof that the introduction of small dissipative forces increases the degree of instability by an amount $s - r_0$ equal to the number of additional frequencies. Hence, in the general case ($s \neq n$), the degree of instability of a gyroscopic system with a singular matrix C when there are small dissipative forces is equal to the degree of static instability.

REFERENCES

1. STRUTT J. B. (LORD RAYLEIGH), *The Theory of Sound*, Vol. 1. Gostekhizdat, Moscow, 1955.
2. COURANT R. and HILBERT D., *Methoden der Matematischen Physik*, Band 1. Springer, Berlin, 1931.
3. ZEVIN A. A., The theory of linear non-conservative systems. *Prikl. Mat. Mekh.* **52**, 3, 386–391, 1988.
4. METELITSYN I. I., The effect of a change in the parameters of linear gyroscopic systems on the oscillation frequencies and damping factors. *Dokl. Akad. Nauk SSSR* **153**, 3, 540–542, 1963.
5. SEIRANYAN A. P., On Metelitsyn's theorems. *Izv. Ross. Akad. Nauk, MTT* **3**, 39–43, 1994.
6. MERKIN D. R., *Gyroscopic Systems*. Nauka, Moscow, 1974.
7. ZHURAVLEV V. F., Extension of Rayleigh's theorem to gyroscopic systems. *Prikl. Mat. Mekh.* **40**, 4, 606–610, 1976.
8. BALINSKII A. I., Behaviour of the frequencies of gyroscopic systems. In *Mathematical Methods and Physico-mechanical Fields*. Naukova Dumka, Kiev, 7, 20–21, 1978.
9. BELLMAN R., *Introduction to Matrix Theory*. Nauka, Moscow, 1969.
10. YAKUBOVICH V. A. and STARZHINSKII V. M., *Linear Differential Equations with Periodic Coefficients and their Applications*. Nauka, Moscow, 1972.
11. KREIN M. G. and YAKUBOVICH V. A., Hamiltonian systems of linear differential equations with periodic coefficients. In *Analytic Methods in the Theory of Non-linear Vibrations: Proceedings of the International Symposium on Non-linear Vibrations*. Izd. Akad. Nauk UkrSSR, Kiev, 1, 277–305, 1963.
12. ZEVIN A. A., Generalization of the Rayleigh theorem to non-linear and parametrically excited systems. *J. Sound and Vibration* **171**, 4, 473–482, 1994.
13. KREIN M. G., Fundamental theorems in the theory of λ-zones of stability of a canonical system of linear differential equations with periodic coefficients. In *In Memory of A. A. Andronov*. Izd. Akad. Nauk SSSR, Moscow, 413–498, 1955.

Translated by E.L.S.